Formulation problem in numerical relativity

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この研究は真貝寿明氏（稲盛）との共同研究です.

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This talk is based on our papers:

review article	gr-qc/0209111 (Nova Science Publ.)
for Ashtekar form.	Phys.Rev.D 60 (1999) 101502, Class.Qaut.Gvav. 17 (2000) 4799
	Class.Qaut.Gvav. 18 (2001) 441
for ADM form.	Phys.Rev.D 63 (2001) 124019, Class.Qaut.Gvav. 19 (2002) 1027
for BSSN form.	Phys.Rev.D 66 (2002) 124003
general	Class.Qaut.Gvav. 20 (2003) L31, Gen.Rel.Grav.36(8)(2004)1931

Outline

- Purpose: Which formulation is suitable for a simulation of Einstein equation?
- Strategy1: Hyperbolic reductions for Einstein equation.
- Strategy2: Constraint propagation analysis gives us an index of stability.

Plan of talks

1. Introduction
2. Hyperbolic reduction
3. Constraint propagation analysis
4. Adjusted systems
5. Summary

1 Introduction

(1) Why is a numerical simulation of Einstein equation necessary?

$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad (\mu, \nu = 0, 1, 2, 3)$	metric on 4 dimensional Manifold
$\Gamma^\mu_{\nu\rho} = (1/2)g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})$	Christoffel symbol (connection)
$R_{\mu\nu} = \partial_\nu \Gamma^\rho_{\mu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\tau_{\mu\rho}\Gamma^\rho_{\tau\nu} - \Gamma^\tau_{\mu\nu}\Gamma^\rho_{\rho\tau}$	Ricci tensor (curvature)
$T_{\mu\nu}$	energy momentum tensor (stress tensor)
$R_{\mu\nu} - (1/2)Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$	Einstein equation
	$(R = R_{\mu\nu}g^{\mu\nu},\ \Lambda =$ cosmological constant$)$

Einstein equation is second rank, non-linear, 10-simultaneous, partial differential equation. It is difficult to get its exact solution without symmetry, In particular dynamical solutions are difficult to get. Then we need to use numerical simulation of Einstein equation.

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$T_{\mu\nu}$ — energy momentum tensor (stress tensor)

$R_{\mu\nu} - (1/2) R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$ — Einstein equation

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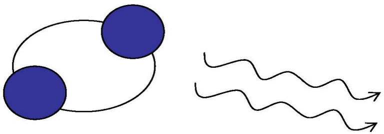
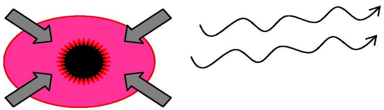




(2) Most traditional formulation: ADM formulation

We have to decompose 4 dimensional Einstein equation into 1 dimension of time and 3 dimensions of space to do numerical simulation. The following is the ADM formulation, which is the most traditional one of spacetime decomposition of Einstein equation.

$$
\begin{array}{ll}
ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) & (i,j = 1,2,3) \\
\quad \alpha : \text{lapse}, \ \beta^i : \text{shift}, \ \gamma_{ij} : \text{spatial metric} & \text{decomposition of metric} \\
K_{ij} := -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - \nabla_i \beta_j - \nabla_j \beta_i) & \text{extrinsic curvature} \\
\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij} - 16\pi\rho - 2\Lambda = 0 & \text{Hamiltonian constraint equation} \\
\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K - 8\pi J_i = 0 & \text{Momentum constraint equation} \\
\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i & \text{evolution equation 1} \\
\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha & +(\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
\qquad -\alpha \Lambda \gamma_{ij} - 8\pi\alpha S_{ij} - 4\pi\alpha\gamma_{ij}(\rho - S^l{}_l) & \text{evolution equation 2}
\end{array}
$$

To do numerical simulation, we first solve the constraint equations on initial spatial surface. And, we decide the gauge function (lapse and shift), and evolve to next spatial surface by using evolution equations. Then the constraint equations are preserved during evolution analytically. But numerically, they increase a little and diverge finally. This is the big problem.

1 Introduction (3) Various fomulations: Ashtekar, BSSN

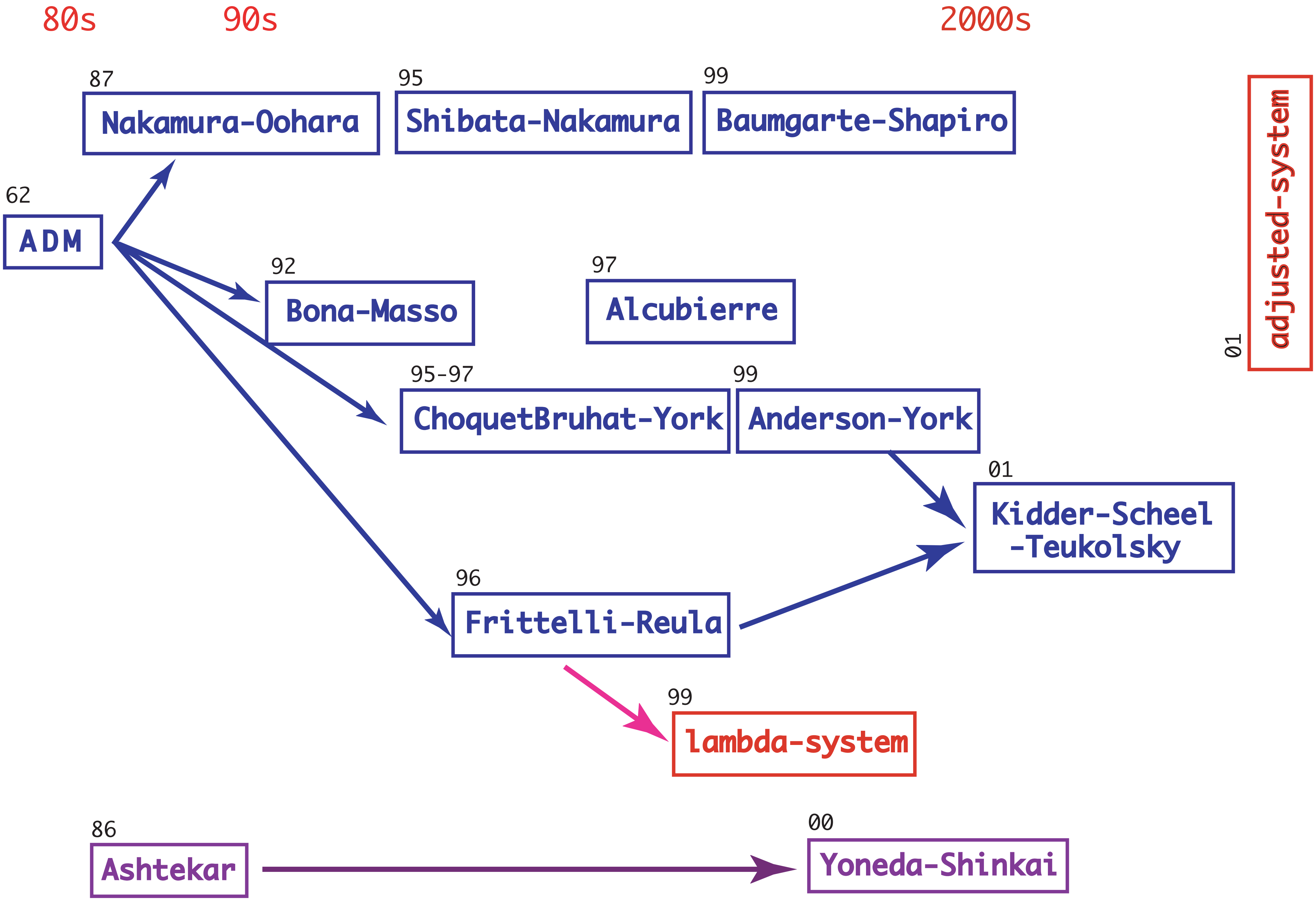
Ashtekar's formulation (Phys.Rev.Lett. **57**, 2244 (1986))

$\tilde{E}^i_a, \mathcal{A}^a_i$ $(i=1,2,3)$, $(a=(1),(2),(3), \mathrm{SO}(3)$ index$)$	canonical pair (densitized triad, Senn connection)
$\underset{\sim}{N}, N^i, \mathcal{A}^a_0$	gauge functions (densitized lapse, shift, triad lapse)
$F^a_{ij} := \partial_i \mathcal{A}^a_j - \partial_j \mathcal{A}^a_i - i\epsilon^a{}_{bc}\mathcal{A}^b_i\mathcal{A}^c_j)$	curvature
$\frac{i}{2}\epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - \Lambda \det \tilde{E} = 0,\ -F^a_{ij}\tilde{E}^j_a = 0,\ \mathcal{D}_i \tilde{E}^i_a = 0,$	constraint equations 1,2,3
$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c,$	evolution equation 1
$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0$	evolution equation 2

BSSN formulation (Pys.Rev.D 52, 5428 (1995))

$\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i$	dynamical variables
$R^{BSSN} + K^2 - K_{ij}K^{ij} - 2\Lambda = 0,\ D_j K^j{}_i - D_i K = 0,$	constraint equations 1,2
$\tilde{\Gamma}^i - \tilde{\Gamma}^i{}_{jk}\tilde{\gamma}^{jk} = 0,\ \det(\tilde{\gamma}_{ij}) = 1,\ \tilde{A}_{ij}\tilde{\gamma}^{ij} = 0$	constraint equations 3,4,5
$\partial_t \varphi = -(1/6)\alpha K + (1/6)\beta^i(\partial_i \varphi) + (\partial_i \beta^i),$	evolution equation 1
$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j \beta^k) + \tilde{\gamma}_{jk}(\partial_i \beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k \beta^k) + \beta^k(\partial_k \tilde{\gamma}_{ij}),$	evolution equation 2
$\partial_t K = -D^i D_i \alpha + \alpha \tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K),$	evolution equation 3
$\partial_t \tilde{A}_{ij} = -e^{-4\varphi}(D_i D_j \alpha)^{TF} + e^{-4\varphi}\alpha(R^{BSSN}_{ij})^{TF} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j + \cdots$	evolution equation 4
$\partial^B_t \tilde{\Gamma}^i = -2(\partial_j \alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j \varphi)) + \cdots$	evolution equation 5

Other various formulations can be thought by arrangement of variables and by adding constraint terms on evolution equations (adjustment).
Which formulation is suitable for a simulation of Einstein equation? (formulation problem)



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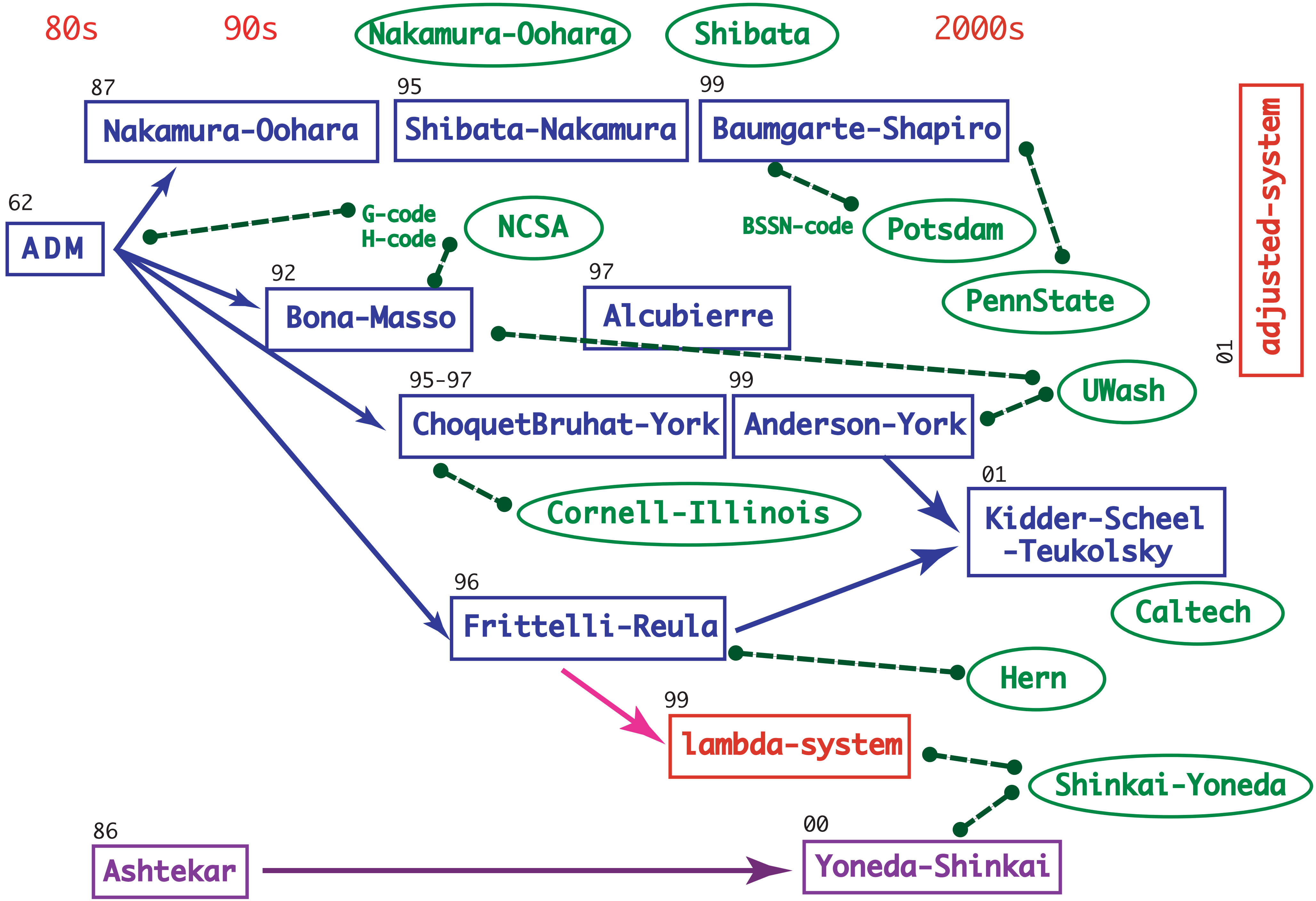

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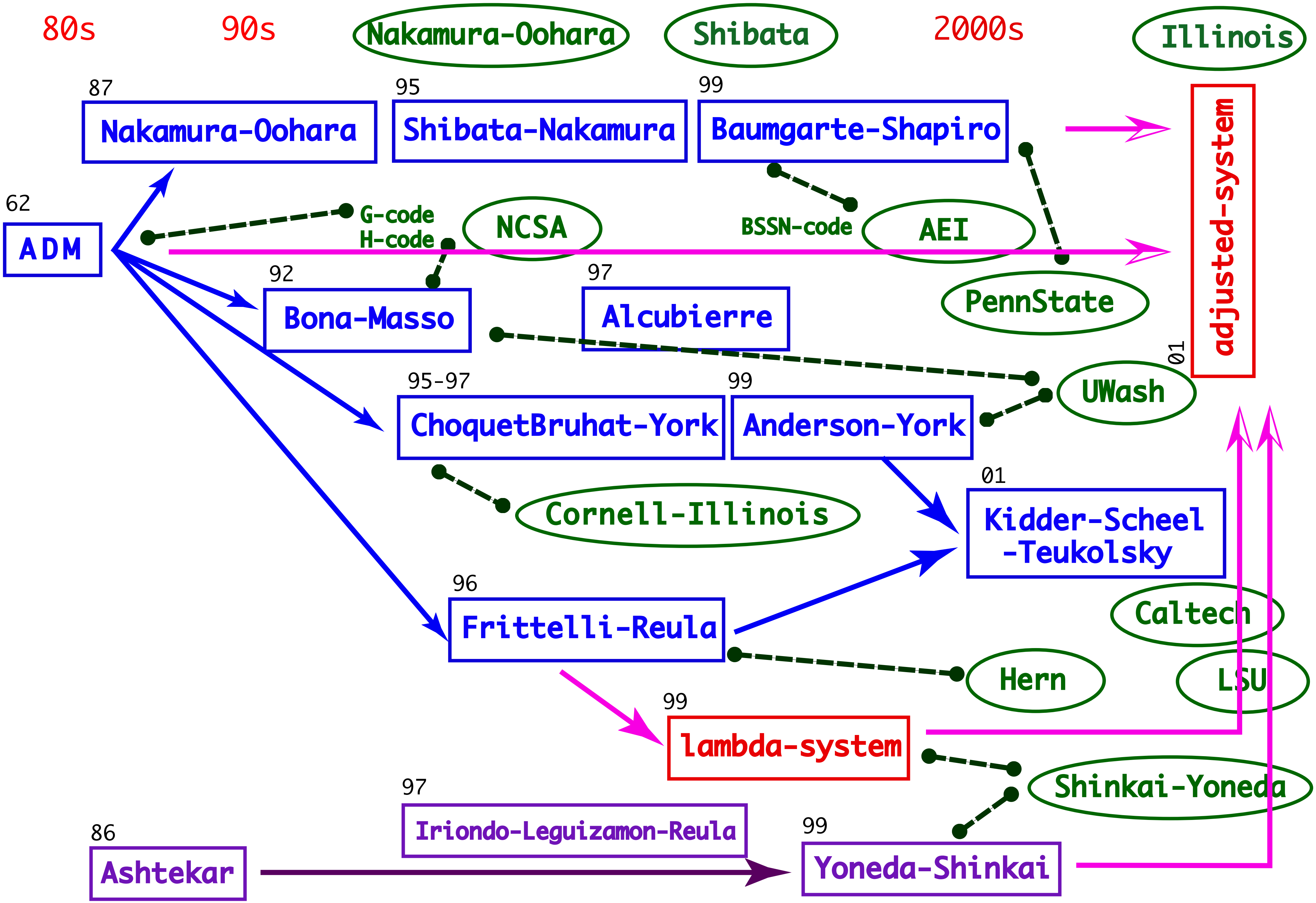
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２階の偏微分方程式の分類

$$a\partial_t^2 u + b\partial_t\partial_x u + c\partial_x^2 u + d\partial_t u + e\partial_x u + f = 0$$

1. $b^2 - 4ac > 0 \Leftrightarrow \partial_t^2 u = \partial_x^2 u \Leftrightarrow$ 双曲型 (hyperbolic): 波動方程式，移流方程式
2. $b^2 - 4ac = 0 \Leftrightarrow \partial_t u = \partial_x^2 u \Leftrightarrow$ 放物型 (parabolic): 熱伝導方程式，拡散方程式
3. $b^2 - 4ac < 0 \Leftrightarrow \partial_t^2 u \; - \partial_x^2 u \Leftrightarrow$ 楕円型 (elliptic): ラプラス方程式，ポアソン方程式

連立方程式へ　$\Box\vec{u} = \vec{0}$ となればもちろん双曲型であるが，そんな簡単にはならない．

１階へ　$\partial_t^2 u = \partial_x^2 u \Leftrightarrow v_t := \partial_t u,\, v_x := \partial_x u$ と置けば１階化できる．（対称行列が前に出る）

$$\partial_t \begin{pmatrix} u_t \\ u_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_t \\ u_x \end{pmatrix}$$

１階双曲型の分類

$$\partial_t \vec{u} = A\partial_x \vec{u} + \vec{b}(\text{non differential terms})$$

A が対称行列（複素ならエルミート行列） $\Leftrightarrow$ 対称双曲型 (symmetric hyperbolic)

A が実対角化可能 $\Leftrightarrow$ 強双曲型 (strongly hyperbolic)

A の固有値が全て実 $\Leftrightarrow$ 弱双曲型 (weakly hyperbolic)

初期値（境界値）問題の適切性 (well-posedness)

与えられた初期値に対して，時間発展後の解を求めよ.

1. 解が存在 する
2. 解が一意的である
3. 解が連続的である (初期値の微小摂動に対し，解の摂動も微小)

対称双曲型とエネルギー不等式

$$\partial_t \vec{u}(t,x) = A(x)\partial_x \vec{u}(t,x) \quad ({}^tA = A)$$

$$|\vec{u}(t)|^2 = \int dx\ ({}^t\vec{u}(t,x))\vec{u}(t,x)$$

$$\partial_t|\vec{u}(t)|^2 = \int dx \left\{({}^t\partial_t\vec{u})\vec{u} + ({}^t\vec{u})(\partial_t\vec{u})\right\} = \int dx \left\{{}^t(A\partial_x\vec{u})\vec{u} + ({}^t\vec{u})(A\partial_x\vec{u})\right\}$$

$$= \int dx \left\{({}^t\partial_x\vec{u})A\vec{u} + ({}^t\vec{u})A\partial_x\vec{u}\right\} = \int dx \left\{-({}^t\vec{u})\partial_x(A\vec{u}) + ({}^t\vec{u})A\partial_x\vec{u}\right\}$$

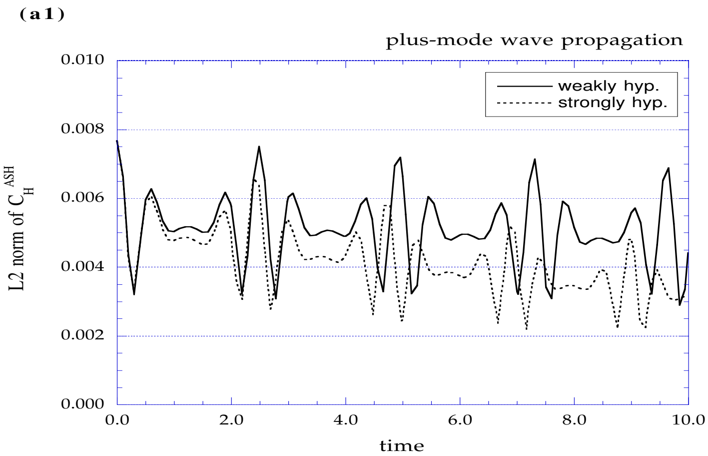
$$= \int dx \left\{-({}^t\vec{u})(\partial_x A)\vec{u})\right\} \le \gamma|\vec{u}(t)|^2 \qquad (\gamma = |-\partial_x A| = \sup_{\vec{v}} \frac{|-\partial_x A\vec{v}|}{|\vec{v}|})$$

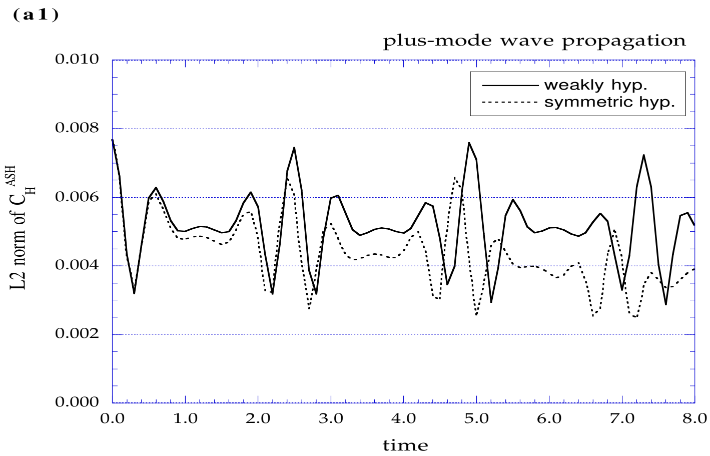
$$|\vec{u}(t)|^2 \le \exp(\gamma t)|\vec{u}(0)|^2$$

2 Hyperbolic reduction

We apply a formulation which reveals 1st order hyperbolicity. It is expected that *wellposed behavior*, *better boundary treatment* (by information of propagation speed) and *known numerical techniques in Newtonian dynamics*. There are many try of hyperbolic reductions of Einstein equation. I give an example from GY-HS Phys. Rev. Lett. 82(1999), 263-266

- Ashtekar formulation is weakly hyperbolic (principal matrix has real spectrum) one.
- strongly hyperbolic (principal matrix is real diagonalizable) when $\mathcal{A}_0^a = \mathcal{A}_i^a N^i$, metric reality and adjusting $(N^i \delta_{ab} + i\underset{\sim}{N} \epsilon_{ab}{}^c \tilde{E}_c^i)\mathcal{C}_G^b$ to $\partial_t \tilde{E}_a^i$, $e^{-2}\underset{\sim}{N}\tilde{E}_i^a \mathcal{C}_H - ie^{-2}\underset{\sim}{N}\epsilon^{abc}\tilde{E}_{bi}\tilde{E}_c^j \mathcal{C}_{Mj}$ to $\partial_t \mathcal{A}_i^a$
- symmetric hyperbolic (principal matrix is Hermite) when $\mathcal{A}_0^a = \mathcal{A}_i^a N^i, \partial_i N = 0$, triad reality and above adjustment





Are hyperbolic formulations actually helpful in numerical simulations?
Unfortunately, we do not have conclusive answer to it yet.

2 Hyperbolic reduction

Theoretical issues

- Well-posedness of non-linear hyperbolic formulations is obtained only locally in time domain.
- Energy inequality indicates exponential boundedness of norm which does not forbid divergence
- The discussion of hyperbolicity only uses characteristic part of evolution equations, and ignore the non-characteristic part.

Numerical issues

- Earlier numerical comparisons reported the advantages of hyperbolic formulations, but they were against to the standard ADM formulation. [Cornell-Illinois, NCSA, ...]
- If the gauge functions are evolved with hyperbolic equations, then their finite propagation speeds may cause a pathological shock formation [Alcubierre].
- Some group [HS-GY, Hern] reported no drastic numerical differences between three hyperbolic levels, while other group [Calabresse,Cornell-Caltech] reported that strongly hyperbolic is good and weakly hyperbolic is bad. Of course, these statements only cast on a particular formulations and models to apply.

Proposed symmetric hyperbolic systems were not always the best one for numerics.

3 Constraint propagation analysis

For time evolution systems with constraints in general

$$\begin{aligned} \partial_t u^a &= f(u^a, \partial u^a, \partial\partial u^a) && \text{evolution equations} \\ C^\alpha &= C^\alpha(u^a, \partial u^a, \partial\partial u^a) \approx 0 && \text{constraints} \end{aligned}$$

If constraints are first class, constraint propagation takes this form

$$\partial_t C^\alpha = A_0 C^\alpha + A_1 \partial C^\alpha + A_2 \partial\partial C^\alpha + \cdots \quad \textbf{constraint propagation}$$

Analytically, constraints are satisfied during evolution. But numerically, does not. By Fourier transformation, we rewrite constraint propagation with each modes, which is ODE.

$$\begin{aligned} \partial_t \hat{C}^\alpha &= A_0 \hat{C}^\alpha + A_1 (i\vec{k}) \hat{C}^\alpha + A_2 (i\vec{k})(i\vec{k}) \hat{C}^\alpha + \cdots \\ &= \underbrace{\left(A_0 + A_1 (i\vec{k}) + A_2 (i\vec{k})(i\vec{k}) + \cdots\right)}_{M} \hat{C}^\alpha \quad \text{constraint propagation 2} \end{aligned}$$

constraint propagation matrix

we substitute background metric into $M \to M_{bg}$

$$\text{CAF} := \text{Eigenvalues } M_{bg} \quad \textbf{Constraint Amplification Factors (CAF)}$$

By evaluating CAFs before simulations, we will be able to predict constraint violation in numerical evolution.

A Classification of Constraint Propagations (cont.)

$\partial_t C = \lambda C \Rightarrow C = C(0)\exp(\lambda t)$

(C1) Asymptotically **constrained** : (Violation of constraints converges to zero.)
$\approx$ all the real part of CAFs are **negative**

(C2) Asymptotically **bounded** : (Violation of constraints is bounded at a certain value.)
$\approx$ all the real part of CAFs are **non-positive**

(C3) **Diverge**: (At least one constraint will diverge.)
$\approx$ there exists CAF with **positive** real part

A Classification of Constraint Propagations (cont.)

$\partial_t C = MC$, CAF = Eigenvalues(M)

(C1) Asymptotically **constrained** : (Violation of constraints decays.)
$\Leftrightarrow$ all the real part of CAFs are **negative**

(C2) Asymptotically **bounded** : (Violation of constraints is bounded at a certain value.)
$\Leftrightarrow$ all the real part of CAFs are **non-positive**
and **Jordan** matrices for eigenvalues with zero real part are **diagonal**
$\Leftarrow$ all the real part CAFs are non-positive and M is **diagonalizable**

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or there exists non diagonal **Jordan** matrix for eigenvalues with zero real part

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Each eigenvalue evaluation.
Real part: Negative is better than zero and positive is worst.
Imaginary part: non-zero is better than zero for avoiding degeneracy.

Example1: Maxwell equation

$$\begin{aligned}
\partial_t E^i &= -c\epsilon^i{}_j{}^l\ \partial_l B^j,\ \partial_t B^i = c\epsilon^i{}_j{}^l\ \partial_l E^j \\
C_E &:= \partial_i E^i \approx 0, ,\ C_B := \partial_i B^i \approx 0, \\
\partial_t C_E &= 0,\ \partial_t C_B = 0 \\
\text{CAF} &= (0,0) \quad (\text{asymptotically bounded})
\end{aligned}$$

Example 2: ADM equation

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \\
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij} K^{ij}, \\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K, \\
\partial_t \mathcal{H} &= \beta^j (\partial_j \mathcal{H}) - 2\alpha \gamma^{ji} (\partial_i \mathcal{M}_j) + 2\alpha K \mathcal{H} + \alpha (\partial_l \gamma_{mn})(2\gamma^{ml}\gamma^{nj} - \gamma^{mn}\gamma^{lj}) \mathcal{M}_j - 4\gamma^{im}(\partial_m \alpha) \mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) + \beta^j (\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - (\partial_i \alpha)\mathcal{H} - \beta^k \gamma^{jm} (\partial_i \gamma_{mk}) \mathcal{M}_j + (\partial_i \beta_m) \gamma^{mj} \mathcal{M}_j. \\
\text{CAF} &= (0, 0, \pm\sqrt{-k^2}) \quad (\text{in Minkowskii background}) \quad (\text{asymptotically bounded})
\end{aligned}$$

Example 3: BSSN

$$
\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i), \\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}), \\
\partial_t^B K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), \\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R_{ij}^{BSSN})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j + (\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} \\
&\quad -(2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}), \\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi)) - \partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k\beta^i) \\
&\quad -\tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)). \\
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, \\
\mathcal{M}_i^{BSSN} &= \nabla_j K^j{}_i - \nabla_i K \\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} \\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij} \\
\mathcal{S} &= \tilde{\gamma} - 1 \\
\text{CAF} &= (0\,(\times 3), \pm\sqrt{-k^2}\,\text{(3 pairs)}) \quad \text{(in Minkowskii background)} \quad (\text{asymptotically bounded})
\end{aligned}
$$

4 adjusted system

$$
\begin{aligned}
& \text{Add constraint terms to evolution equations (adjust)} \\
\partial_t u^a &= f(u^a, \partial u^a, \partial\partial u^a) + F(C^\alpha, \partial C^\alpha, \partial\partial C^\alpha) \\
& \text{constraint propagation changes depending on them, too} \\
\partial_t C^\alpha &= A_0 C^\alpha + A_1 \partial C^\alpha + A_2 \partial\partial C^\alpha + \cdots + B_0 C^\alpha + B_1 \partial C^\alpha + B_2 \partial\partial C^\alpha + \cdots \\
& \text{CAF changes depending on them, too}
\end{aligned}
$$

We should adjust so that CAFs improve.

Advantage of adjusted system

1. Available even if the base system is not a symmetric hyperbolic.
2. Keep the number of the variables same with the original system.
3. Unified understanding for formulation problem is possible using the notions of adjustment and CAF

Example 1: adjusted Maxwell equations

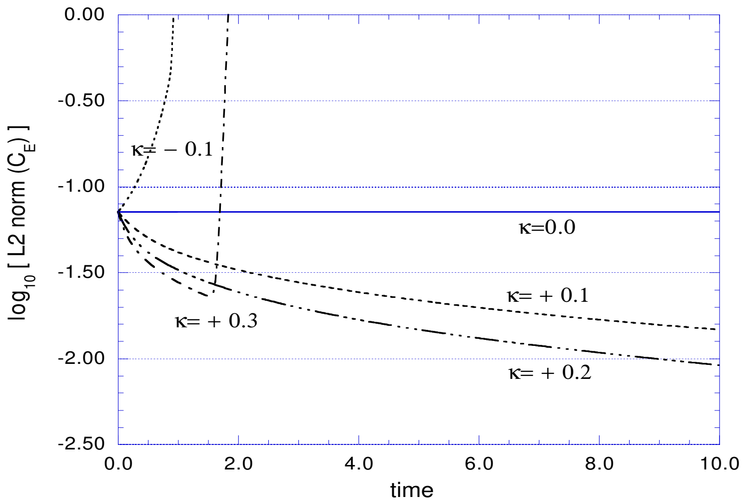
$$\partial_t E_i = \epsilon_i{}^{jk}\partial_j B_k + \kappa\partial_j C_E,\ \partial_t B_i = -\epsilon_i{}^{jk}\partial_j E_k + \kappa\partial_j C_B$$ evolution equations

$$C_E = \mathrm{div}\, E = 0,\ C_B = \mathrm{div}\, B = 0$$ constraint equations

$$\partial_t \begin{pmatrix} \tilde{C}_E \\ \tilde{C}_B \end{pmatrix} = \begin{pmatrix} -\kappa|\vec{k}|^2 & 0 \\ 0 & -\kappa|\vec{k}|^2 \end{pmatrix} \begin{pmatrix} \tilde{C}_E \\ \tilde{C}_B \end{pmatrix}$$ constraint propagation

$$\mathrm{CAF} = (-\kappa|\vec{k}|^2, -\kappa|\vec{k}|^2)$$

CAF is negative when $\kappa > 0$



Example 2: adjusted ADM formulations (Detweiler type adjustment)

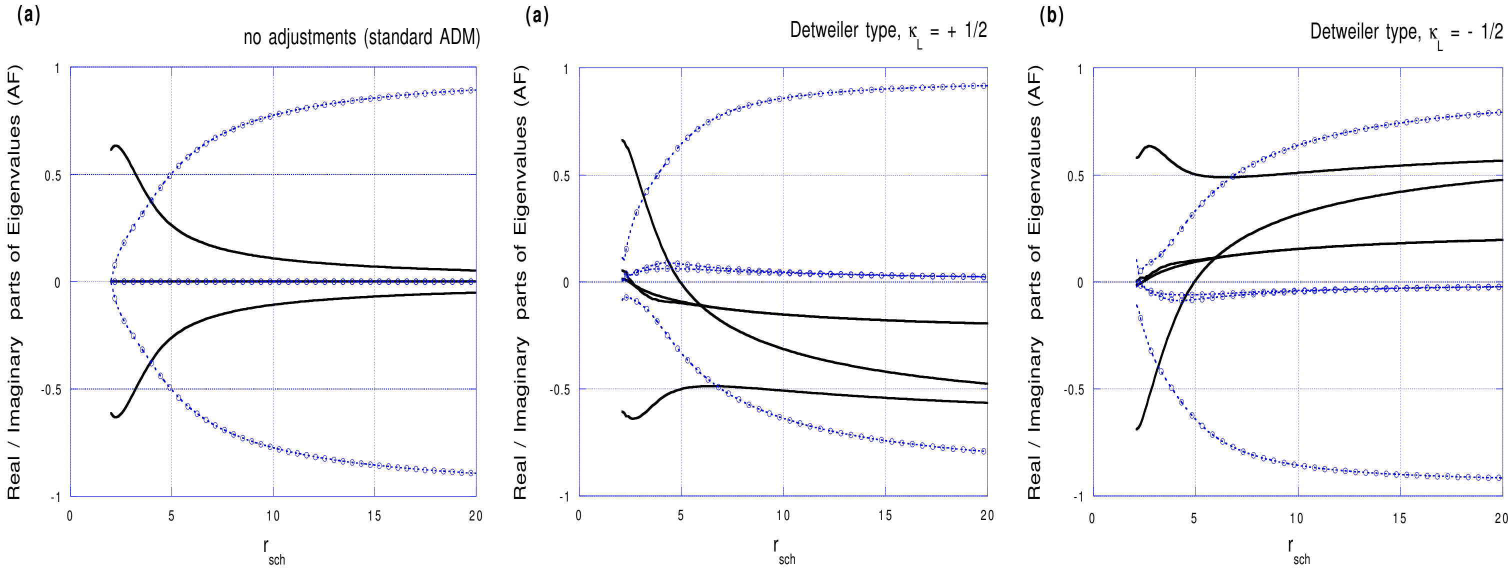
$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+\kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^3 (3 \partial_{(i} \alpha \delta^k_{j)} - \partial_l \alpha \gamma_{ij} \gamma^{kl}) \mathcal{M}_k$$
$$+\kappa_L \alpha^3 \delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}) \nabla_k \mathcal{M}_l$$

In case of Minkowskii background, CAF becomes

$$\text{CAF} = (-(1/2)\kappa_L |\vec{k}|, -(1/2)\kappa_L |\vec{k}|, -(4/3)\kappa_L |\vec{k}| \pm |\vec{k}| \sqrt{-1 + (4/9)\kappa_L^2 |\vec{k}|^2}))$$

In case of Schwarzschild background, CAF becomes



Example 2: adjusted ADM formulations (numerical test)

1. **original ADM** $(\partial_t K_{ij} = \text{original term} + \alpha\gamma_{ij}\mathcal{H})$ CAF=(0,0,0,0) (diverge)

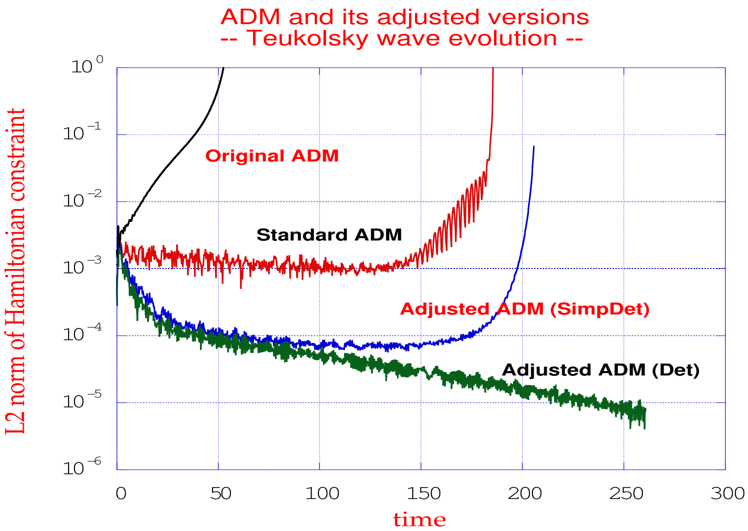
2. **standard ADM** (no adjust) $\text{CAF} = (0, 0, \pm\sqrt{-|\vec{k}|^2}) = (0, 0, \Im m, \Im m)$

3. **simplified Detweiler type** $(\partial_t \gamma_{ij} = \text{original term} - \kappa\alpha\gamma_{ij}\mathcal{H})$
   $\text{CAF} = (0, 0, -\kappa|\vec{k}|^2 \pm |\vec{k}|\sqrt{-1 + \kappa|\vec{k}|^2}) = (0, 0, -, -)$

4. **Detweiler type** $\partial_t \gamma_{ij} = \text{original term} - \kappa_L \alpha^3 \gamma_{ij}\mathcal{H}$

$$\partial_t K_{ij} = \text{original term} + \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij})\mathcal{H} + \kappa_L \alpha^3 (3\partial_{(i}\alpha\delta^k_{j)} - \partial_l \alpha \gamma_{ij}\gamma^{kl})\mathcal{M}_k + \kappa_L \alpha^3 \delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl})\nabla_k \mathcal{M}_l$$

$$\text{CAF} = (-(1/2)\kappa_L|\vec{k}|, -(1/2)\kappa_L|\vec{k}|, -(4/3)\kappa_L|\vec{k}| \pm |\vec{k}|\sqrt{-1 + (4/9)\kappa_L^2|\vec{k}|^2})) = (-, -, -, -)$$



Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{1}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{2}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{3}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{4}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{5}$$

Adjustments in evolution equations

$$\partial_t^B \varphi = \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j \mathcal{S})\beta^j, \tag{6}$$

$$\partial_t^B \tilde{\gamma}_{ij} = \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k \mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{7}$$

$$\partial_t^B K = \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{8}$$

$$\begin{aligned}\partial_t^B \tilde{A}_{ij} = {} & \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + ((1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij}) - (1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k \\ & +\alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k)\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_t^B \tilde{\Gamma}^i = {} & \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} - (2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) \\ & +2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) + (1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) \\ & +(5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) + (1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}).\end{aligned} \tag{10}$$

Effect of adjustments

No.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	diag?	Constr. Amp. Factors in Minkowskii background
0.	standard ADM	use	use	-	-	-	yes	$(0, 0, \Im, \Im)$
1.	BSSN no adjustment	use	use	use	use	use	yes	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$ Growing modes
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
8.	ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$	use+adj	use+adj	-	-	-	no	$(0, 0, 0, 0)$
9.	ignore $\mathcal{G}^i$, $\mathcal{A}$	use+adj	use+adj	use+adj	-	-	yes	$(0, \Im, \Im, \Im, \Im, \Im, \Im)$
10.	ignore $\mathcal{G}^i$	use+adj	use+adj	-	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0)$
11.	ignore $\mathcal{A}$	use+adj	use+adj	use+adj	-	use+adj	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
12.	ignore $\mathcal{S}$	use+adj	use+adj	use+adj	use+adj	-	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$

New Proposals :: Improved (adjusted) BSSN systems

TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t \tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$
\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{A}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{A}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{A}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{A}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}
$$

or in the flat background

$$
\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{A}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{A}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{A}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{A}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}
$$

Constraint Amplification Factors with each adjustment

adjustment		CAFs	diag?	effect of the adjustment	
$\partial_t \phi$	$\kappa_{\phi\mathcal{H}}\, \alpha \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}} k^2)$	no	$\kappa_{\phi\mathcal{H}} < 0$ makes 1 Neg.	
$\partial_t \phi$	$\kappa_{\phi\mathcal{G}}\, \alpha \tilde{D}_k \mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}} < 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{SD}\, \alpha \tilde{\gamma}_{ij} \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), (3/2)\kappa_{SD} k^2)$	yes	$\kappa_{SD} < 0$ makes 1 Neg.	Case (B)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\, \alpha \tilde{\gamma}_{ij} \tilde{D}_k \mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\, \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k$	(0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2} \pm \sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2} < 0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2} k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2} > 0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\, \alpha \tilde{\gamma}^{jk} (\tilde{D}_j \mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2)$, $(1/3)\kappa_{K\mathcal{M}} k^2 \pm (1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}} < 0$ makes 2 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\, \alpha \tilde{\gamma}_{ij} (\tilde{D}^k \mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1} k^2)$	yes	$\kappa_{A\mathcal{M}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\, \alpha (\tilde{D}_{(i} \mathcal{M}_{j)})$	(0,0, $-k^2\kappa_{A\mathcal{M}2}/4 \pm \sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2} > 0$ makes 7 Neg	Case (D)
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2} k^2)$	yes	$\kappa_{A\mathcal{A}2} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\, \alpha \tilde{D}^i \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2} k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\, \alpha \mathcal{G}^i$	$(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1} < 0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\, \alpha \tilde{D}^j \tilde{D}_j \mathcal{G}^i$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2} > 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\, \alpha \tilde{D}^i \tilde{D}_j \mathcal{G}^j$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3} > 0$ makes 2 Neg. 1 Pos.	

Yoneda-HS, PRD66 (2002) 124003

Example: the Ashtekar equations

HS Yoneda, CQG 17 (2000) 4799

Adjusted dynamical equations:

$$
\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \underbrace{+ X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}}_{adjust} \\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \underbrace{+ Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}}_{adjust}
\end{aligned}
$$

Adjusted and linearized:

$$
X = Y = Z = 0,\ P^{ia}_b = \kappa_1(iN^i\delta^a_b),\ Q^a_i = \kappa_2(e^{-2}\underset{\sim}{N}\tilde{E}^a_i),\ R^{aj}{}_i = \kappa_3(-ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c)
$$

Fourier transform and extract 0th order of the characteristic matrix:

$$
\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \end{pmatrix} = \begin{pmatrix} 0 & i(1+2\kappa_3)k_j & 0 \\ i(1-2\kappa_2)k_i & \kappa_3 \epsilon^{kj}{}_i k_k & 0 \\ 0 & 2\kappa_3 \delta^j_a & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \end{pmatrix}
$$

Eigenvalues:

$$
\left(0, 0, 0, \pm\kappa_3\sqrt{-kx^2 - ky^2 - kz^2}, \pm\sqrt{(-1+2\kappa_2)(1+2\kappa_3)(kx^2+ky^2+kz^2)}\right)
$$

In order to obtain non-positive real eigenvalues:

$$
(-1+2\kappa_2)(1+2\kappa_3) < 0
$$

Time reversal symmetry(TRS) breaking adjustment

$\boxed{\text{TRS}}$: Time reversal($t \to -t$) に対し形を変えない
TR に対し，偶の parity $\alpha, \gamma_{ij}, \partial_x, \nabla_i, R^{(3)}_{ij}, \mathcal{H}$
TR に対し，奇の parity $\beta^i, K_{ij}, \partial_t, \mathcal{M}_i$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \qquad (i, j = 1, 2, 3)$$

α : lapse, β^i : shift, γ_{ij} : spatial metric — decomposition of metric

$$K_{ij} := -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - \nabla_i \beta_j - \nabla_j \beta_i) \qquad \text{extrinsic curvature}$$

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij} - 16\pi\rho - 2\Lambda = 0 \qquad \text{Hamiltonian constraint equation}$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K - 8\pi J_i = 0 \qquad \text{Momentum constraint equation}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \qquad \text{evolution equation 1}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$-\alpha\Lambda\gamma_{ij} - 8\pi\alpha S_{ij} - 4\pi\alpha\gamma_{ij}(\rho - S^l{}_l) \qquad \text{evolution equation 2}$$

Einstein eq. は TRS を持っている．

$\boxed{\text{TRS を持った背景，TRS を持った座標を使う限り，CAF の符号は偏ることはない．}}$

TRS を破るような補正をすると，CAF の符号が偏る．例えば

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i + \underbrace{P_{ij}\mathcal{H}}_{\text{TRS 破る}} + \underbrace{Q_{ij}{}^k \mathcal{M}_k}_{\text{TRS 守る}} \qquad (P, Q \text{ は定数})$$

5 Discussion

5.1 Application 1 : Constraint Propagation in $N+1$ dim. space-time

HS-Yoneda, submitted to GRG (2003)

Dynamical equation has N-dependency

Only the matter term in $\partial_t K_{ij}$ has N-dependency.

$$
\begin{aligned}
0 \approx \mathcal{C}_H &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu = \frac{1}{2}({}^{(N)}R + K^2 - K^{ij}K_{ij}) - 8\pi\rho_H - \Lambda, \\
0 \approx \mathcal{C}_{Mi} &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp^\nu_i = D_j K^j_i - D_i K - 8\pi J_i, \\
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \\
\partial_t K_{ij} &= \alpha {}^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell{}_j K_{i\ell} - D_i D_j \alpha \\
&\quad + \beta^k (D_k K_{ij}) + (D_j\beta^k) K_{ik} + (D_i\beta^k) K_{kj} - 8\pi\alpha \left(S_{ij} - \frac{1}{N-1}\gamma_{ij} T \right) - \frac{2\alpha}{N-1}\gamma_{ij}\Lambda,
\end{aligned}
$$

Constraint Propagations remain the same

From the Bianchi identity, $\nabla^\nu \mathcal{S}_{\mu\nu} = 0$ with $\mathcal{S}_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}$, we get

$$
\begin{aligned}
0 = n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\mu Y_\mu + Y_\nu n^\mu \nabla_\mu n^\nu - 2Y_\mu n_\nu (\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu(\nabla^\mu X), \\
0 = h_i{}^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= \nabla^\mu Z_{i\mu} + Y_i(\nabla^\mu n_\mu) + Y_\mu(\nabla^\mu n_i) + X(\nabla^\mu n_i) n_\mu + n_\mu(\nabla^\mu Y_i).
\end{aligned}
$$

- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ with $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow$ matter eq.
- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, \mathcal{C}_H, \mathcal{C}_{Mi}, \kappa\gamma_{ij}\mathcal{C}_H)$ with $\nabla^\mu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0 \Rightarrow$ CP eq.

5.2 Application 2 : Constraint Propagation of Maxwell field in Curved space

HS-Yoneda, in preparation

Towards a robust GR-MHD system:

- **Maxwell eqs in curved space-time**

$$\begin{aligned}
\partial_t E^i &= \epsilon^{ijk} D_j(\alpha B_k) - 4\pi\alpha J^i + \alpha K E^i + \pounds_\beta E^i \\
\partial_t B^i &= -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i + \pounds_\beta B^i \\
\mathcal{C}_E &:= D_i E^i - 4\pi\rho_e \\
\mathcal{C}_B &:= D_i B^i
\end{aligned}$$

- **CP of Maxwell system in curved space-time**

$$\begin{aligned}
\partial_t C_E &= \alpha K C_E + \beta^j D_j C_E \\
\partial_t C_B &= \alpha K C_B + \beta^j D_j C_B
\end{aligned}$$

- **CP of ADM+Maxwell**

$$\partial_t \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix}$$

- **CP of ADM+Maxwell+Hydro**
  in progress.

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\qquad \partial_t u = J\partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\qquad \partial_t \lambda = \alpha C - \beta\lambda$ $\quad (\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]

Example: Maxwell adjusted (cont.)

GY HS, CQG 18 (2001) 441

$$
\begin{aligned}
\partial_t E^i &= -c\epsilon^i{}_j{}^l\, \partial_l B^j + \kappa\partial^i C_E, \;\; \partial_t B^i = c\epsilon^i{}_j{}^l\, \partial_l E^j + \kappa\partial^i C_B \\
C_E &:= \partial_i E^i \approx 0, , \;\; C_B := \partial_i B^i \approx 0, \\
\partial_t C_E &= \kappa\partial_i\partial^i C_E, \;\; \partial_t C_B = \kappa\partial_i\partial^i C_B \\
\text{CAF} &= (-\kappa, -\kappa)
\end{aligned}
$$

original Maxwell($\kappa = 0$), CAF= $(0, 0)$, averaged VN factor for FTCS=1.013

adjusted Maxwell($\kappa = 0.1$), CAF= $(-0.1, -0.1)$, averaged VN factor for FTCS=0.8017

adjusted Maxwell($\kappa = -0.1$), CAF= $(+0.1, +0.1)$, averaged VN factor for FTCS=1.2246

Adjustments also reduce the von Neumann factors.

In other words, adjustment is just like adding viscosity terms to evolution equations.

5 Summary

- Towards a stable and accurate numerical relativity, which formulation is suitable?
- Hyperbolic reduction is one of strategy but not perfect.
- Constraint propagation analysis gives us an index of stability.
- If we adjust so that CAFs improve, numerical error is decreased.

Future

- dynamical control of adjustment
- constraint propagation analysis without substitution of background
- apply constraint propagation analysis to the study of stability of gauge conditions and coordinate
- apply some technique of hydro simulation to numerical relativity

FAQ

Q1 Why does CAFs=zero indicate the divergence of the system?

It happens. See this simple example. $\partial_t \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$

Though the eigenvalues are all zero, but c_2 =constant, c_1 is constantly increasing.

Q2 Why do we need to substitute the background metric for evaluating CAFs?

There are two reasons. First reason is because it is too complicated without substitution. Second reason is because a sign of eigenvalue does not often clear without substitution.

Q3 Does the prior evaluation by CAFs predict the numerical stability perfectly?

Unfortunately No. Because it is an approximate evaluation, we cannot prevent the numerical divergence of error when it appears.

Q4 What is the greatest advantage of this proposal of CAFs ?

CAFs enables us to evaluate the system's stability before we start a numerical simulation. Positive CAFs surely indicate the divergence of the simulation. Negative CAFs surely indicate that constraint manifold is the attractor.

Q5 Does it get closer to a true solution really?

When there is a exact solution, I can compare numerical solution with it and it has been checked by some examples. When there is not a exact solution, I can only check whether evolution and constraint always satisfy enough.