Hyperbolic systems of Einstein equation in the Ashtekar formulation

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目的

Einstein equationに関しての

1. **hyperbolic reduction**
2. 初期値問題，初期境界値問題の
   well-posedness
3. 数値相対論の
   stability

なぜ難しいか？

- 準線型連立
- 時間軸の未決定（ゲージの自由度）

双曲型運動方程式

u_α に対する 1 階の quasi-linear system

$$\partial_t u_\alpha = J^{l\beta}{}_\alpha(u)\partial_l u_\beta + K_\alpha(u)$$

(I). J^l の固有値が全部実: **weakly hyp.**

(II). J^l が実対角化可能: **strongly hyp.**

(III). J^l がエルミート行列: **symmetric hyp.**

Einstein eq. $(\mu, \nu = t, x, y, z)$

$$R_{\mu\nu} = 0$$

ADM 形式 $(i, j = x, y, z)$

equation of motion

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2NK_{ij} + D_j N_i + D_i N_j, \\
\partial_t K_{ij} &= N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N \\
&\quad + (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij}
\end{aligned}$$

constraint

$$\begin{aligned}
\mathcal{C}_H^{\mathrm{ADM}} &:= {}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} \approx 0 \\
\mathcal{C}_{Mi}^{\mathrm{ADM}} &:= D_j(K^{ij} - \gamma^{ij}\mathrm{tr}K) \approx 0
\end{aligned}$$

hyperbolic reduction by
Bona-Masso, ChoquetBruhat-York-Anderson,
Frittelli-Reula, Friedrich, ...

Ashtekar 形式 $(i, j = x, y, z,\ a, b = 1, 2, 3)$

A. Ashtekar, Phys.Rev.Lett.**57**, 2244 (1986).
triad:$\tilde{E}^i_a$ と conection:$\mathcal{A}^a_i$ から成る正準形式

equation of motion

$$\begin{aligned} \partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c, \\ \partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 \end{aligned}$$

constraint

$$\mathcal{C}^{\rm Ash}_H = \frac{i}{2}\epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} \approx 0,\ \mathcal{C}^{\rm Ash}_{Mi} = -F^a_{ij}\tilde{E}^j_a \approx 0,$$

$\mathcal{C}^{\rm Ash}_{Ga} = \mathcal{D}_i \tilde{E}^i_a \approx 0$ where $F^a_{ij} := \partial_i \mathcal{A}^a_j - \partial_j \mathcal{A}^a_i - i\epsilon^a{}_{bc}\mathcal{A}^b_i \mathcal{A}^c_j$

metric reality

(primary) $\mathbf{Im}(\tilde{E}^i_a \tilde{E}^j_a) = 0$

(secondary) $W^{ij} := \mathbf{Re}(\epsilon^{abc}\tilde{E}^k_a \tilde{E}^{(i}_b \mathcal{D}_k \tilde{E}^{j)}_c) = 0$

triad reality

(primary) $\mathbf{Im}(\tilde{E}^i_a) = 0$

(secondary) $W^{ij} = 0$ and

$\mathbf{Re}(\mathcal{A}^a_0) = (\partial_i \underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2}e^{-2}\tilde{E}^b_i \underset{\sim}{N} \tilde{E}^{ja}\partial_j \tilde{E}^i_b + N^i \mathbf{Re}(\mathcal{A}^a_i)$

Ashtekar で hyperbolic reduction

metric reality を課す

- (Ia) そのままで weakly hyp.
  固有値 $= \{0\,(6), N^l\,(4), N^l \pm N\sqrt{\gamma^{ll}}\,(4)\}$
- (IIa) strong hyp になる必要十分条件は
  $N^l \neq 0$ nor $\pm N\sqrt{\gamma^{ll}}$
  固有値は (Ia) と同じ

triad reality を課す

- (Ib) triad reality の secondary condition:
  $\mathrm{Re}(\mathcal{A}_0^a) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2}e^{-2}\tilde{E}_i^b \underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}_b^i + N^i\mathrm{Re}(\mathcal{A}_i^a)$
  $= (\partial_i N)E^{ia} + N^i\mathrm{Re}(\mathcal{A}_i^a)$ により 2 階に.
  $\partial_i N = 0, \mathcal{A}_0^a = \mathcal{A}_i^a N^i$ と仮定して weakly hyp.
  固有値 $= \{0\,(3),\ N^l\,(7),\ N^l \pm N\sqrt{\gamma^{ll}}\,(4)\}$

運動方程式の右辺を constraint で補正

- (IIIa) symmetric hyp. を目指すと triad reality が必要 (ex. エルミート性: $\tilde{E}^i_a - \bar{\tilde{E}}^i_a = 0$)
  $\partial_i N = 0$, $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$ を仮定
  constraint 補正係数が**一意的**に求められる
  補正 term to $\partial_t \tilde{E}^i_a$
  $= (N^i \delta_{ab} + i\underset{\sim}{N} \epsilon_{ab}{}^c \tilde{E}^i_c) \mathcal{C}^b_G$
  補正 term to $\partial_t \mathcal{A}^a_i$
  $= e^{-2} \underset{\sim}{N} \tilde{E}^a_i \mathcal{C}_H - i e^{-2} \underset{\sim}{N} \epsilon^{abc} \tilde{E}_{bi} \tilde{E}^j_c \mathcal{C}_{Mj}$
  固有値 $= \{N^l$ (6), $N^l \pm \sqrt{\gamma^{ll}} N$ (6)$\}$
- (IIb) 上と同じ補正で，metric reality を課すと strongly hyp. 固有値も上と同じ

hyperbolic system	Eqs of motion	reality condition	gauge conditions required
Ia	original	metric	-
Ib	original	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$
IIa	original	metric	$N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$
IIb	modified	metric	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$
IIIa	modified	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$

出来たこと

Ashtekar形式をベースにhyperbolic reduction
実数条件，ゲージ条件，補正を調べた
hypのlevelが上がると，ゲージ条件が厳しく

課題1

特に有用なsymmetric hyp.ではゲージ条件が強いので，なんとかして弱められないか
→ 難航中

課題2

実際にどのlevelのhypが数値計算で有利か？
→ (pre-print) Hyperbolic formulations and numerical relativity: experiments based on Ashtekar's connection variables

Further

Ashtekar hyperbolic systemをベースにasymptotic constrained system
→ (pre-print) Asymptotically constrained system: experiments based on Ashtekar's formulation